Nitche method for computer homogenization boundary conditions

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Abstract

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Keywords Nitche method, Multiscale, Computational homogenization.

1 Constrains

1

$$\mathcal{R} = \mathcal{C}^{\mathrm{T}}(\mathcal{C}\mathcal{C}^{\mathrm{T}})^{-1} \tag{1}$$

2

$$\mathcal{P} = \mathcal{R}\mathcal{C} \tag{2}$$

3

$$\mathcal{Q} = \mathcal{I} - \mathcal{P} \tag{3}$$

4

$$\mathcal{P} = \mathcal{P}\mathcal{P} \tag{4}$$

2 Problem formulation

7

$$\mathbf{t}(\mathbf{u}) = -\frac{1}{\gamma}\mathcal{R}(\mathcal{C}\mathbf{u} - \mathbf{g} - \gamma\mathcal{C}\mathbf{t}(\mathbf{u})) \tag{5}$$

8

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma} \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathbf{P}\mathbf{v}\mathrm{d}\Gamma = 0 \tag{6}$$

9

$$\mathbf{v} = \mathcal{R}(\mathcal{C}\mathbf{v} - \phi\gamma\mathcal{C}\mathbf{t}(\mathbf{v})) + \phi\gamma\mathcal{P}\mathbf{t}(\mathbf{v}) + \mathcal{Q}\mathbf{v} \tag{7}$$

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10

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
&-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\phi\gamma\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
&-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{R}(\mathcal{C}\mathbf{v}-\phi\gamma\mathcal{C}\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
&=0
\end{aligned}
\tag{8}
$$

11

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
&-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\phi\gamma\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
&+\int_\Gamma \frac{1}{\gamma}(\mathcal{R}(\mathcal{C}\mathbf{u}-\mathbf{g}-\gamma\mathcal{C}\mathbf{t}(\mathbf{u})))^{\mathrm{T}}\mathcal{R}(\mathcal{C}\mathbf{v}-\phi\gamma\mathcal{C}\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
&=0
\end{aligned}
\tag{9}
$$

12

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
&-\int_\Gamma \phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
&+\int_\Gamma \frac{1}{\gamma}(\mathcal{R}(\mathcal{C}\mathbf{u}-\mathbf{g}-\gamma\mathcal{C}\mathbf{t}(\mathbf{u})))^{\mathrm{T}}\mathcal{R}(\mathcal{C}\mathbf{v}-\phi\gamma\mathcal{C}\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
&=0
\end{aligned}
\tag{10}
$$

13

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
&-\int_\Gamma \phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
&+\int_\Gamma \frac{1}{\gamma}(\mathbf{u}-\gamma\mathbf{t}(\mathbf{u}))^{\mathrm{T}}\mathcal{P}(\mathbf{v}-\phi\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
&-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}(\mathbf{v}-\phi\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
&=0
\end{aligned}
\tag{11}
$$

14

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v})\\
&-\int_\Gamma \phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma\\
&+\int_\Gamma \frac{1}{\gamma}\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{v}-\phi\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{t}(\mathbf{v})-\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{v}+\phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma\\
&-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}(\mathbf{v}-\phi\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma\\
&=0
\end{aligned}
\tag{12}
$$

15

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v})\\
&-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{v}\mathrm{d}\Gamma\\
&-\int_\Gamma \phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma\\
&+\int_\Gamma \frac{1}{\gamma}\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{v}-\phi\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{t}(\mathbf{v})+\phi\gamma\mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma\\
&-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}(\mathbf{v}-\phi\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma\\
&=0
\end{aligned}
\tag{13}
$$

17

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v})\\
&-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{v}\mathrm{d}\Gamma\\
&+\int_\Gamma \frac{1}{\gamma}\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{v}-\phi\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma\\
&-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}(\mathbf{v}-\phi\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma\\
&=0
\end{aligned}
\tag{14}
$$

18

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) \\
-\int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{v}\mathrm{d}\Gamma \\
+\int_\Gamma \frac{1}{\gamma}\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{v} - \phi\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}\mathbf{v} - \phi\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
= 0
\end{aligned} \tag{15}$$

19

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) - \int_\Gamma \mathbf{t}^{\mathrm{T}}(\mathbf{u})\mathcal{P}\mathbf{v}\mathrm{d}\Gamma + \int_\Gamma \frac{1}{\gamma}\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{v}\mathrm{d}\Gamma - \int_\Gamma \phi\mathbf{u}^{\mathrm{T}}\mathcal{P}\mathbf{t}(\mathbf{v})\mathrm{d}\Gamma \\
-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}\mathbf{v}\mathrm{d}\Gamma + \int_\Gamma \phi\mathbf{g}^{\mathrm{T}}\mathcal{R}^{\mathrm{T}}\gamma\mathbf{t}(\mathbf{v}))\mathrm{d}\Gamma \\
= 0
\end{aligned} \tag{16}$$

3 Periodic BC

$$\mathcal{C} = [1, -1] \tag{17}$$

$$\mathcal{R} = \frac{1}{2}\left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} \tag{18}$$

$$\mathcal{P} = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right] \tag{19}$$

20

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) - \int_\Gamma [\mathbf{t}_+^{\mathrm{T}}(\mathbf{u})\, \mathbf{t}_-^{\mathrm{T}}(\mathbf{u}] \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} \mathbf{v}_+ \\ \mathbf{v}_- \end{array} \right\} \mathrm{d}\Gamma \\
+\int_\Gamma \frac{1}{\gamma}[\mathbf{u}_+^{\mathrm{T}}\, \mathbf{u}_-^{\mathrm{T}}] \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} \mathbf{v}_+ \\ \mathbf{v}_- \end{array} \right\} \mathrm{d}\Gamma \\
-\int_\Gamma \phi[\mathbf{u}_+^{\mathrm{T}}\, \mathbf{u}_-^{\mathrm{T}}] \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} \mathbf{t}_+(\mathbf{v}) \\ \mathbf{t}_-(\mathbf{v}) \end{array} \right\} \mathrm{d}\Gamma \\
-\int_\Gamma \frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}[1\ -1] \left\{ \begin{array}{c} \mathbf{v}_+ \\ \mathbf{v}_- \end{array} \right\} \mathrm{d}\Gamma \\
+\int_\Gamma \phi\mathbf{g}^{\mathrm{T}}[1\ -1] \left\{ \begin{array}{c} \mathbf{t}_+(\mathbf{v}) \\ \mathbf{t}_-(\mathbf{v}) \end{array} \right\} \mathrm{d}\Gamma \\
= 0
\end{aligned} \tag{20}$$

21

$$
\begin{aligned}
& a(\mathbf{u},\mathbf{v}) \\
-\int_{\Gamma_+}(\mathbf{t}_+^{\mathrm{T}}(\mathbf{u})-\mathbf{t}_-^{\mathrm{T}}(\mathbf{u}))\mathbf{v}_+\mathrm{d}\Gamma-\int_{\Gamma_-}(\mathbf{t}_-^{\mathrm{T}}(\mathbf{u})-\mathbf{t}_+^{\mathrm{T}}(\mathbf{u}))\mathbf{v}_-\mathrm{d}\Gamma & \\
+\int_{\Gamma_+}\frac{1}{\gamma}(\mathbf{u}_+^{\mathrm{T}}-\mathbf{u}_-^{\mathrm{T}})\mathbf{v}_+\mathrm{d}\Gamma+\int_{\Gamma_-}\frac{1}{\gamma}(\mathbf{u}_-^{\mathrm{T}}-\mathbf{u}_+^{\mathrm{T}})\mathbf{v}_-\mathrm{d}\Gamma & \\
-\int_{\Gamma_+}\phi(\mathbf{u}_+^{\mathrm{T}}-\mathbf{u}_-^{\mathrm{T}})\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma-\int_{\Gamma_-}\phi(\mathbf{u}_-^{\mathrm{T}}-\mathbf{u}_+^{\mathrm{T}})\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma & \\
-\int_{\Gamma_+}\frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathbf{v}_+\mathrm{d}\Gamma+\int_{\Gamma_-}\frac{1}{\gamma}\mathbf{g}^{\mathrm{T}}\mathbf{v}_-\mathrm{d}\Gamma & \\
+\int_{\Gamma_+}\phi\mathbf{g}^{\mathrm{T}}\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma-\int_{\Gamma_-}\phi\mathbf{g}^{\mathrm{T}}\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma & \\
&=0
\end{aligned}
\tag{21}
$$

$$
\mathbf{D}=\frac{1}{2}\begin{bmatrix} 2x & 0 & 0 & y & 0 & z \\ 0 & 2y & 0 & x & z & 0 \\ 0 & 0 & 2z & z & 0 & x \end{bmatrix}
\tag{22}
$$

$$
\mathbf{g}=\mathcal{P}\mathbf{D}\boldsymbol{\varepsilon}=(\mathbf{D}_+-\mathbf{D}_-)\boldsymbol{\varepsilon}
\tag{23}
$$

21

$$
\begin{aligned}
& a(\mathbf{u},\mathbf{v}) \\
-\int_{\Gamma_+}(\mathbf{t}_+^{\mathrm{T}}(\mathbf{u})-\mathbf{t}_-^{\mathrm{T}}(\mathbf{u}))\mathbf{v}_+\mathrm{d}\Gamma-\int_{\Gamma_-}(\mathbf{t}_-^{\mathrm{T}}(\mathbf{u})-\mathbf{t}_+^{\mathrm{T}}(\mathbf{u}))\mathbf{v}_-\mathrm{d}\Gamma & \\
+\int_{\Gamma_+}\frac{1}{\gamma}(\mathbf{u}_+^{\mathrm{T}}-\mathbf{u}_-^{\mathrm{T}})\mathbf{v}_+\mathrm{d}\Gamma+\int_{\Gamma_-}\frac{1}{\gamma}(\mathbf{u}_-^{\mathrm{T}}-\mathbf{u}_+^{\mathrm{T}})\mathbf{v}_-\mathrm{d}\Gamma & \\
-\int_{\Gamma_+}\phi(\mathbf{u}_+^{\mathrm{T}}-\mathbf{u}_-^{\mathrm{T}})\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma-\int_{\Gamma_-}\phi(\mathbf{u}_-^{\mathrm{T}}-\mathbf{u}_+^{\mathrm{T}})\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma & \\
-\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+}\frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_+\mathrm{d}\Gamma-\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-}\frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_-\mathrm{d}\Gamma & \\
+\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+}\phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma+\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-}\phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma & \\
&=0
\end{aligned}
\tag{24}
$$

22

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
-\int_{\Gamma_+} \mathbf{t}_+^{\mathrm{T}}(\mathbf{v}_+ - \mathbf{v}_-)\mathrm{d}\Gamma - \int_{\Gamma_-} \mathbf{t}_-^{\mathrm{T}}(\mathbf{v}_- - \mathbf{v}_+)\mathrm{d}\Gamma \\
+\int_{\Gamma_+} \frac{1}{\gamma}(\mathbf{u}_+^{\mathrm{T}} - \mathbf{u}_-^{\mathrm{T}})\mathbf{v}_+\mathrm{d}\Gamma + \int_{\Gamma_-} \frac{1}{\gamma}(\mathbf{u}_-^{\mathrm{T}} - \mathbf{u}_+^{\mathrm{T}})\mathbf{v}_-\mathrm{d}\Gamma \\
-\int_{\Gamma_+} \phi(\mathbf{u}_+^{\mathrm{T}} - \mathbf{u}_-^{\mathrm{T}})\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma - \int_{\Gamma_-} \phi(\mathbf{u}_-^{\mathrm{T}} - \mathbf{u}_+^{\mathrm{T}})\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma \\
-\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+} \frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_+\mathrm{d}\Gamma - \boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-} \frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_-\mathrm{d}\Gamma \\
+\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+} \phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma + \boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-} \phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma \\
&= 0
\end{aligned}
\tag{25}
$$

23

$$
\begin{aligned}
&a(\mathbf{u},\mathbf{v}) \\
-\int_{\Gamma_+} \mathbf{t}_+^{\mathrm{T}}(\mathbf{v}_+ - (1-\epsilon)\mathbf{v}_-)\mathrm{d}\Gamma - \int_{\Gamma_-} \mathbf{t}_-^{\mathrm{T}}(\mathbf{v}_- - (1-\epsilon)\mathbf{v}_+)\mathrm{d}\Gamma \\
+\int_{\Gamma_+} \frac{1}{\gamma}(\mathbf{u}_+^{\mathrm{T}} - (1-\epsilon)\mathbf{u}_-^{\mathrm{T}})\mathbf{v}_+\mathrm{d}\Gamma + \int_{\Gamma_-} \frac{1}{\gamma}(\mathbf{u}_-^{\mathrm{T}} - (1-\epsilon)\mathbf{u}_+^{\mathrm{T}})\mathbf{v}_-\mathrm{d}\Gamma \\
-\int_{\Gamma_+} \phi(\mathbf{u}_+^{\mathrm{T}} - (1-\epsilon)\mathbf{u}_-^{\mathrm{T}})\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma - \int_{\Gamma_-} \phi(\mathbf{u}_-^{\mathrm{T}} - (1-\epsilon)\mathbf{u}_+^{\mathrm{T}})\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma \\
-\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+} \frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_+\mathrm{d}\Gamma - \boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-} \frac{1}{\gamma}\mathbf{D}^{\mathrm{T}}\mathbf{v}_-\mathrm{d}\Gamma \\
+\boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_+} \phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_+(\mathbf{v})\mathrm{d}\Gamma + \boldsymbol{\varepsilon}^{\mathrm{T}}\int_{\Gamma_-} \phi\mathbf{D}^{\mathrm{T}}\mathbf{t}_-(\mathbf{v})\mathrm{d}\Gamma \\
&= 0
\end{aligned}
\tag{26}
$$

$$
\mathbf{g} = (\mathbf{D}_+ - (1-\epsilon)\mathbf{D}_-)\boldsymbol{\varepsilon} \tag{27}
$$